

Right-angled Coxeter groups with n -dimensional Sierpiński compacta as boundaries.

Jacek Świątkowski *

Instytut Matematyczny, Uniwersytet Wrocławski
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
e-mail: swiatkow@math.uni.wroc.pl

Abstract. For arbitrary positive integer n , we describe a large class of right-angled Coxeter systems (W, S) for which the visual boundary $\partial_\infty(W, S)$ is homeomorphic to the n -dimensional Sierpiński compactum. We also provide a necessary and sufficient condition for a planar simplicial complex L under which the right angled Coxeter system (W, S) whose nerve is L has the visual boundary $\partial_\infty(W, S)$ homeomorphic to the Sierpiński curve.

1. Introduction.

The *boundary at infinity* (or the *visual boundary*) of a Coxeter system (W, S) is a compact metric space canonically associated to the system, denoted $\partial_\infty(W, S)$. It reflects some aspect of the large-scale behaviour of the associated Coxeter group W , and in case when W is word-hyperbolic, it coincides with the Gromov boundary ∂W of W . (We refer the reader to Chapter 12 in [Da] for a detailed exposition of this concept.) In this paper, for arbitrary positive integer n , we describe, a large class of right-angled Coxeter systems (W, S) for which the boundary $\partial_\infty(W, S)$ is homeomorphic to the n -dimensional Sierpiński compactum.

1.1 Definition. The n -dimensional Sierpiński compactum is a compact metric space described, uniquely up to homeomorphism, as any subspace Π of the sphere S^{n+1} satisfying the following conditions:

- (1) the complement of Π is dense in S^{n+1} ;
- (2) each connected component U of $S^{n+1} \setminus \Pi$ is an open $(n+1)$ -cell in S^{n+1} , i.e. the pair (S^{n+1}, U) is homeomorphic to the standard pair $(S^{n+1}, \text{int}(B^{n+1}))$, where B^{n+1} denotes here the hemisphere in S^{n+1} ;
- (3) the family \mathcal{U} of all connected components of the complement $S^{n+1} \setminus \Pi$ is null, i.e. for any $\epsilon > 0$ only finitely many of these components have diameter greater than ϵ ;
- (4) closures \overline{U} of the components $U \in \mathcal{U}$ are pairwise disjoint.

The uniqueness property in the above definition was proved by Cannon [Ca], see also Theorem 7.2.7 in [DV]. The set of conditions appearing in this definition is called the *positional characterization* of the n -dimensional Sierpiński compactum. The 1-dimensional Sierpiński compactum coincides with the space better known as Sierpiński curve or Sierpiński carpet.

We now explain the special terminology appearing in the statement of the main result of the paper, Theorem 1.3 below.

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1.2 Definition (sphere with holes). An $(n + 1)$ -sphere with holes is a simplicial complex L equipped with a PL embedding into the PL $(n + 1)$ -sphere S^{n+1} satisfying the following properties:

- (0) L is a proper subspace of S^{n+1} ;
- (1) L is flag;
- (2) for any connected component Ω of $S^{n+1} \setminus L$, denoting its closure in S^{n+1} by $\overline{\Omega}$, the pair $(S^{n+1}, \overline{\Omega})$ is PL homeomorphic to the standard pair (S^{n+1}, B^{n+1}) , where the topological boundary $\text{bd}(\Omega) = \overline{\Omega} \setminus \Omega$ corresponds to the boundary sphere ∂B^{n+1} ;
- (3) the boundary $\text{bd}(\Omega)$ of any component Ω of $S^{n+1} \setminus L$ is a full subcomplex of L .

The term *3-convexity* appearing below in the statement of Theorem 1.3 is explained in Section 4 (see Definition 4.1). In the same section we show that for any positive integer n there are many simplicial complexes L to which the theorem applies.

1.3 Theorem. Let (W, S) be a right-angled Coxeter system whose nerve L is an $(n + 1)$ -sphere with holes. Suppose also that the following two conditions hold:

- (1) for any two distinct connected components of $S^{n+1} \setminus L$, their closures in S^{n+1} are either disjoint, or intersect at a single simplex of L ;
 - (2) the boundary $\text{bd}(\Omega)$ of any component Ω of $S^{n+1} \setminus L$ is a 3-convex subcomplex of L .
- Then the visual boundary $\partial_\infty(W, S)$ is homeomorphic to the n -dimensional Sierpiński compactum.

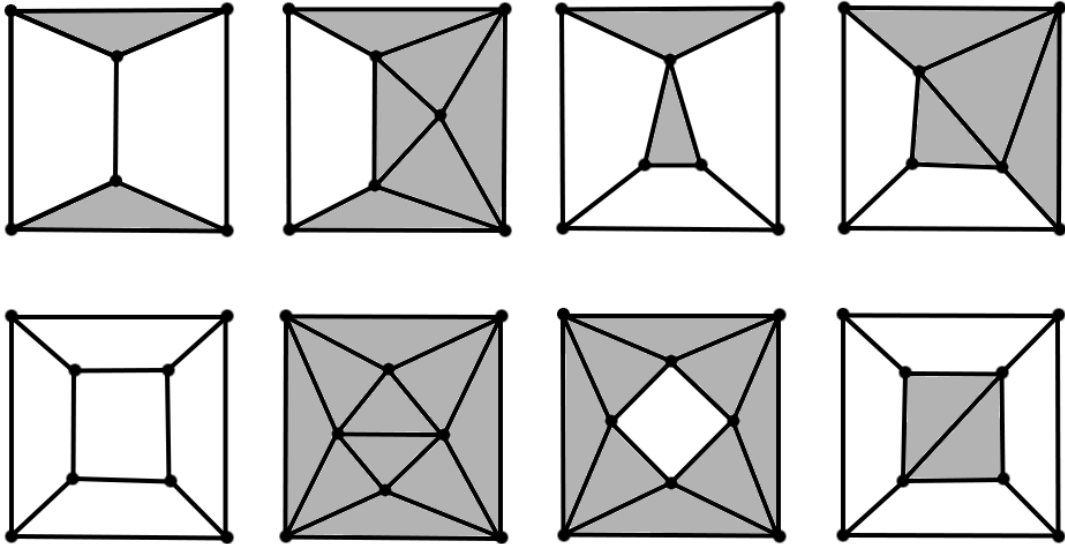


Figure 1. Examples of nerves of non-hyperbolic right-angled Coxeter groups with Sierpiński carpet boundaries.

In the case $n = 1$, Theorem 1.3 can be used to extend Theorem 1 of [Sw2] (which characterizes those right-angled word-hyperbolic Coxeter groups with planar nerves whose Gromov boundary is homeomorphic to the Sierpiński curve) to non-hyperbolic case. More precisely, we call a simplicial complex *unseparable* if it is connected, has no separating

simplex, no separating pair of nonadjacent vertices, and no separating full subcomplex isomorphic to the (simplicial) suspension of a simplex. Theorem 1.3 yields then the following.

1.4 Corollary. *Let (W, S) be a right angled Coxeter system with planar nerve L . Then the boundary $\partial_\infty(W, S)$ is homeomorphic to the Sierpiński curve if and only if L is unseparable and distinct from a simplex and from a triangulation of S^2 .*

Examples of some nerves of non-hyperbolic right-angled Coxeter systems with Sierpiński carpet boundaries are presented at Figure 1. The "only if" part of the corollary is provided by Lemma 1.1 in [Sw2]. Thus, in order to deduce the corollary from Theorem 1.3, it is sufficient to note that if a planar flag simplicial complex L is unseparable, distinct from a simplex and from a triangulation of S^2 then it is a 2-sphere with holes satisfying conditions (1) and (2) of Theorem 1.3. We omit a straightforward proof of this observation.

1.5 Remark. As we show below (in Lemmas 4.8 and 4.9), examples of Coxeter systems (W, S) to which Theorem 1.3 applies, and for which the corresponding group W is word-hyperbolic, appear only in dimensions $n \in \{1, 2\}$. (Non-hyperbolic examples appear in arbitrary dimension.) This leads to a question, whether there are any right-angled word-hyperbolic Coxeter groups with Gromov boundary homeomorphic to the n -dimensional Sierpiński compactum, for $n \geq 3$. This question seems to remain open.

The paper is organized as follows. Sections 2 and 6 contain essential arguments for proving Theorem 1.3, and Sections 3–5 present various preparatory results and techniques applied in these essential arguments. More precisely, in Section 2 we present this part of the argument which uses only the assumption that the nerve L of a right-angled Coxeter system (W, S) is an $(n + 1)$ -sphere with holes. Under this restricted assumption, we construct an embedding of $\partial_\infty(W, S)$ into S^{n+1} , and we show that under this embedding the subspace $\partial_\infty(W, S)$ satisfies conditions (1) and (2) of the positional characterization (given in Definition 1.1) of the n -dimensional Sierpiński compactum. In Section 6 we use the full strength of the assumptions of Theorem 1.3, i.e. we assume that the sphere with holes L satisfies also assumptions (1) and (2) from the statement. We then show that, under the embedding into S^{n+1} described in Section 2, the subspace $\partial_\infty(W, S)$ satisfies conditions (3) and (4) of Definition 1.1. This concludes the proof that $\partial_\infty(W, S)$ is homeomorphic to the n -dimensional Sierpiński compactum.

Section 3 contains a preparatory result of independent interest, namely that the visual boundary of a CAT(0) $(n + 1)$ -dimensional PL manifold with nonempty connected convex boundary is homeomorphic to the n -disk B^n (see Proposition 3.1, which is in fact slightly stronger). This result is essentially used in Section 2, to show that under the described embedding of $\partial_\infty(W, S)$ in S^{n+1} the components of the complement are open $(n + 1)$ -cells in S^{n+1} (which corresponds to condition (2) in Definition 1.1).

In Section 4 we present the concept of 3-convexity, which appears in assumption (2) of the statement of Theorem 1.3. We then derive basic properties of this concept, and discuss existence of nerves L satisfying all the assumptions of Theorem 1.3, for any n .

In Section 5 we consider convex subcomplexes in CAT(0) cubical complexes which satisfy the additional assumption of *local 3-convexity* (which means that the link of the subcomplex at its any vertex is a 3-convex subcomplex in the corresponding vertex link of the whole complex). We then deduce some rather strong global geometric property of

such subcomplexes, which resembles the behaviour of convex subsets in hyperbolic spaces (see Proposition 5.2 for the statement). This global property plays crucial role in our arguments in Section 6. The concept of local 3-convexity in $CAT(0)$ cubical complexes seems to be interesting on its own, and potentially useful for other applications.

In Section 6 (and more precisely in Lemma 6.3) we describe one more technical concept of independent interest, namely a natural family of metrics (compatible with the topology) on the visual boundary $\partial_\infty X$ of a complete $CAT(0)$ space X . Surprisingly, these metrics seem to be not widely known. We use one of these metrics for showing that the family of connected components of the complement $S^{n+1} \setminus \partial_\infty(W, S)$ is null (which corresponds to condition (3) in Definition 1.1).

2. Some conclusions for arbitrary $(n + 1)$ -spheres with holes L .

In this section we work under the assumption that the nerve L of a right-angled Coxeter system (W, S) is an $(n + 1)$ -sphere with holes (i.e. we ignore conditions (1) and (2) from the statement of Theorem 1.3). We show the following result, which is a first step in the proof of Theorem 1.3. (Note that assertions (1) and (2) below correspond to conditions (2) and (1) in Definition 1.1, respectively.)

2.1 Proposition. *Let (W, S) be a right-angled Coxeter system whose nerve L is an $(n+1)$ -sphere with holes. Then*

- (1) *there is an embedding of $\partial_\infty(W, S)$ in S^{n+1} such that the connected components of its complement are open $(n + 1)$ -cells in S^{n+1} ;*
- (2) *topological dimension of the boundary satisfies $\dim(\partial_\infty(W, S)) = n$, and hence for any embedding of $\partial_\infty(W, S)$ in S^{n+1} the complement is dense in S^{n+1} .*

To get assertion (1) above, we first describe and analyze below, after quick review of the general geometric background, a specific embedding of $\partial_\infty(W, S)$ in S^{n+1} . (We will use this embedding not only here, but also in the later sections of the paper, in other parts of the proof of Theorem 1.3.) We also make essential use of a technical result, Proposition 3.1, whose statement and proof are presented in the next section. A short proof of assertion (2) is given at the end of this section.

Throughout this paper we will often work with $CAT(0)$ geodesic metric spaces, as described e.g. in [BH]. Given a complete $CAT(0)$ space E , we denote by $\partial_\infty E$ the visual boundary (at infinity) of E , as defined e.g. in Chapter II.8 in [BH]. We note that if $F \subset E$ is a closed convex subspace of a complete $CAT(0)$ space, then F is also complete and $CAT(0)$, and the boundary $\partial_\infty F$ is canonically a subset in the boundary $\partial_\infty E$.

Most of the $CAT(0)$ spaces under our interest will be cubical complexes. Recall that a cubical complex X is $CAT(0)$ (for the standard piecewise euclidean metric d_X) iff X is connected, simply connected, and its every vertex link $Lk(v, X)$ is a flag simplicial complex (cf. Theorem 5.20 on p. 212 in [BH]). A subcomplex Y of a $CAT(0)$ cubical complex X is convex (in the sense that for any two points of Y the geodesic in X connecting these points is contained in Y) iff Y is connected and for each vertex $v \in Y$ the link $Lk(v, Y)$ is a full subcomplex in the corresponding link $Lk(v, X)$.

Recall also that for any right-angled Coxeter system (W, S) , its nerve L is a flag simplicial complex, and its associated Coxeter-Davis complex Σ is a $CAT(0)$ cubical complex

whose every vertex link is isomorphic to L (see Proposition 7.3.4 and Theorem 12.2.1(i) in [Da]). If W_T is a special subgroup of W corresponding to a subset $T \subset S$, then (W_T, T) is also a right-angled Coxeter system, and its nerve L_T is canonically a full subcomplex of L . Moreover, the Coxeter-Davis complex Σ_T of the system (W_T, T) is canonically a subcomplex in Σ , and for any vertex v of Σ_T the pair of links $(\text{Lk}(v, \Sigma), \text{Lk}(v, \Sigma_T))$ is isomorphic to the pair (L, L_T) . In particular, Σ_T is a convex subcomplex of Σ and its visual boundary $\partial_\infty \Sigma_T$ is a subspace in $\partial_\infty \Sigma$.

Now, under notation and assumptions of Proposition 2.1, we describe a specific embedding of $\partial_\infty(W, S)$ in S^{n+1} . Given an embedding of L in S^{n+1} as in the definition of an $(n+1)$ -sphere with holes, for each connected component Ω of the complement $S^{n+1} \setminus L$ attach to L , along the subcomplex $\text{bd}(\Omega)$ (which is a PL triangulated n -sphere), the simplicial cone over this subcomplex. This yields a flag PL triangulation L' of the sphere S^{n+1} such that L appears in L' as a full subcomplex. Denote by (W', S') the right-angled Coxeter system with nerve L' . Since the boundary of a special subgroup is naturally a subspace in the boundary of a right-angled Coxeter group, we get that $\partial_\infty(W, S) \subset \partial_\infty(W', S')$. Since the latter boundary is homeomorphic to the sphere S^{n+1} (see e.g. Theorem (3b.2) in [DJ] or Corollary 1 in [Dr]), it follows that $\partial_\infty(W, S)$ is embedded in S^{n+1} .

After fixing an embedding $L < L'$ as above, note that if L is a triangulation of the sphere S^n then both assertions of Proposition 2.1 follow from Corollary 3.3 (given in the next section). Thus, in the remaining part of the argument we assume that L is not a triangulation of S^n . In this setting, we introduce some further terminology and notation which will be used both in this section and in the remaining part of the paper. Denote by $\Sigma = \Sigma(W, S)$ and $\Sigma' = \Sigma(W', S')$ the Coxeter-Davis complexes associated to the respective Coxeter systems. We then obviously have $S \subset S'$, $W < W'$ (as a special subgroup) and $\Sigma \subset \Sigma'$ (as a convex subcomplex in a $CAT(0)$ cubical complex). We also have $\partial_\infty(W, S) = \partial_\infty \Sigma \subset \partial_\infty \Sigma' = S^{n+1}$ as the explicit manifestation of the above mentioned embedding of $\partial_\infty(W, S)$ in S^{n+1} .

For any connected component Ω of the complement $L' \setminus L$, let $C = \text{bd}(\Omega)$ be the subcomplex of L' coinciding with the topological boundary of Ω in L' . It follows from our assumptions that C is a PL triangulation of S^n , and the pair (L', C) is (up to PL homeomorphism) the standard pair of PL spheres (S^{n+1}, S^n) . We call each subcomplex C as above a *peripheral cycle* of L in L' . In the now considered case, when L is not a triangulation of S^n , each peripheral cycle is a proper subcomplex of L .

Note that, by condition (3) in Definition 1.2, any peripheral cycle C is a full subcomplex of L . Consequently, it is also a full subcomplex of L' , and thus it describes a special subgroup of both W' and W , which we denote W_C (and the corresponding induced Coxeter system by (W_C, S_C)). Denote by $\Sigma_C = \Sigma(W_C, S_C)$ the Coxeter-Davis complex of the system (W_C, S_C) , and view it canonically as a subcomplex in $\Sigma' = \Sigma(W', S')$. Since C is full both in L' and L , we get that Σ_C is a convex subcomplex of both Σ and Σ' .

Since the pair of nerves (L', C) is the standard pair of PL spheres, locally Σ_C looks in Σ' like an $(n+1)$ -dimensional hyperplane in the euclidean $(n+2)$ -space. Since both Σ_C and Σ' are connected and simply connected (in fact, they are PL homeomorphic to the euclidean spaces E^{n+1} and E^{n+2} , respectively, see Theorem 10.6.1 in [Da]), it follows that Σ_C decomposes Σ' into the union of two subcomplexes H such that $H \setminus \Sigma_C$ is a connected

component of $\Sigma' \setminus \Sigma_C$. We call these subcomplexes the *halfspaces for Σ_C in Σ'* . Note that each such halfspace H is a convex subcomplex of Σ' . This follows from the following two observations:

- a vertex link of H either coincides with the corresponding vertex link of Σ' , or these links form a pair isomorphic to the pair (L', D) , where D is one of the $(n+1)$ -disk subcomplexes in L' bounded by C ;
- any subcomplex D as above is a full subcomplex of L' .

In particular, H is a $CAT(0)$ cubical PL $(n+2)$ -manifold with convex boundary $\partial H = \Sigma_C$.

Next lemma is a direct consequence of Proposition 3.1 (a technical result whose statement and proof are given in the next section).

2.2 Lemma. *If H_0, H_1 are the halfspaces for Σ_C in Σ' then, viewing the boundaries $\partial_\infty H_i, \partial_\infty \Sigma_C$ as subspaces in $\partial_\infty \Sigma' \cong S^{n+1}$, we have:*

- (1) $\partial_\infty H_0 \cap \partial_\infty H_1 = \partial_\infty \Sigma_C$ and $\partial_\infty H_0 \cup \partial_\infty H_1 = \partial_\infty \Sigma'$;
- (2) $\partial_\infty H_i$ are closed $(n+1)$ -cells in $\partial_\infty \Sigma'$ such that $\text{bd}(\partial_\infty H_i) = \partial_\infty \Sigma_C$, where bd denotes the topological boundary in $\partial_\infty \Sigma'$;
- (3) $\partial_\infty H_i \setminus \partial_\infty \Sigma_C$ are the connected components of $\partial_\infty \Sigma' \setminus \partial_\infty \Sigma_C$, and they are open $(n+1)$ -cells in $\partial_\infty \Sigma'$; we also have $\partial_\infty H_i \setminus \partial_\infty \Sigma_C = \partial_\infty \Sigma' \setminus \partial_\infty H_{i+1}$, where $i+1$ is viewed modulo 2.

2.3 Lemma. *Let C be a peripheral cycle of L in L' . Then one of the halfspaces obtained by splitting Σ' along Σ_C contains Σ , and the other intersects Σ only at Σ_C .*

Proof: We first give an alternative description of the halfspaces for Σ_C in Σ' . Recall that the group W' canonically coincides with the vertex set of the cubical complex Σ' , while S' canonically coincides with the vertex set of L' . Let D_C^+ be the subcomplex of L' equal to the $(n+1)$ -disk bounded by C and containing L . Denote also by D_C^- the complementary $(n+1)$ -disk in L' bounded by C . Consider the following condition for a reduced expression $g = s_1 \dots s_n$ of an element $g \in W'$ (where all $s_i \in S'$):

- (*) the first letter s_i not belonging to S_C , if appears at all in the expression, corresponds to a vertex in $D_C^+ \setminus C$.

It follows easily from the Tits' characterization of reduced expressions in Coxeter groups (see Theorem 3.4.3 in [Da]) that property (*) does not depend on the choice of a reduced expression for g , hence it is a condition for g .

Denote by \mathcal{W}_C^+ the set of all $g \in W'$ whose reduced expressions satisfy (*). Denote also by \mathcal{W}_C^- the analogous set for which the disk D_C^+ in condition (*) is replaced with D_C^- . It is not hard to observe (and we leave it without any further justification) that the two halfspaces for Σ_C in Σ' are exactly the subcomplexes of Σ' spanned by the sets \mathcal{W}_C^+ and \mathcal{W}_C^- , viewed as subsets in the vertex set of Σ' . (Here, by the subcomplex spanned by a vertex set we mean the maximal subcomplex having this set as the vertex set.) Accordingly, we denote the two halfspaces by H_C^+ and H_C^- .

Note that obviously we have $W \subset \mathcal{W}_C^+$ and $W \cap \mathcal{W}_C^- = W_C$. Consequently, we get $\Sigma \subset H_C^+$ and $\Sigma \cap H_C^- = \Sigma_C$, which finishes the proof.

In consistency with the proof of Lemma 2.3, we denote by H_C^+ this halfspace for Σ_C in Σ' which contains Σ . We also denote by H_C^- the remaining halfspace. Lemmas 2.2 and 2.3 immediately yield the following.

2.4 Corollary. $\partial_\infty H_C^- \setminus \partial_\infty \Sigma_C = \partial_\infty \Sigma' \setminus \partial_\infty H_C^+$ is a connected component of the complement $\partial_\infty \Sigma' \setminus \partial_\infty \Sigma = S^{n+1} \setminus \partial_\infty(W, S)$. This component is an open $(n+1)$ -cell in S^{n+1} , and its topological boundary coincides with $\partial_\infty \Sigma_C$.

We now pass to describing all other connected components of the complement of $\partial_\infty \Sigma$ in $\partial_\infty \Sigma'$. Note that the group W acts on $\partial_\infty \Sigma'$ so that it preserves the subspace $\partial_\infty \Sigma$. In particular, for any $w \in W$ the set $\partial_\infty \Sigma' \setminus w \cdot \partial_\infty H_C^+ = \partial_\infty \Sigma' \setminus \partial_\infty(w \cdot H_C^+)$ is a connected component of $\partial_\infty \Sigma' \setminus \partial_\infty \Sigma$. Moreover, one easily observes that

- (1) components $\partial_\infty \Sigma' \setminus \partial_\infty(w_i \cdot H_C^+)$, $i = 1, 2$, coincide iff w_1 and w_2 are in the same coset of W/W_C ;
- (2) for distinct peripheral cycles C_1, C_2 of L in L' , and for any $w_1, w_2 \in W$, the components $\partial_\infty \Sigma' \setminus \partial_\infty(w_i \cdot H_{C_i}^+)$ are distinct.

Due to the above observation (1), we denote by $\partial_\infty \Sigma' \setminus \partial_\infty(\bar{w} \cdot H_C^+)$, where $\bar{w} \in W/W_C$ is a coset, the space $\partial_\infty \Sigma' \setminus \partial_\infty(w \cdot H_C^+)$ for any $w \in \bar{w}$. As a consequence of both above observations (1) and (2), the family

$$\partial_\infty \Sigma' \setminus \partial_\infty(\bar{w} \cdot H_C^+) : C \text{ is a peripheral cycle of } L \text{ in } L', \text{ and } \bar{w} \in W/W_C$$

consists of pairwise distinct connected components of $\partial_\infty \Sigma' \setminus \partial_\infty \Sigma$. We will show that in fact there are no other components in this complement.

2.5 Lemma. *Each connected component of $\partial_\infty \Sigma' \setminus \partial_\infty \Sigma$ has a form $U = \partial_\infty \Sigma' \setminus \partial_\infty(\bar{w} \cdot H_C^+)$ for some peripheral cycle C of L in L' , and for some coset $\bar{w} \in W/W_C$.*

Proof: Start with observing that

$$\Sigma = \bigcap_C \bigcap_{\bar{w} \in W/W_C} \bar{w} \cdot H_C^+,$$

where C runs through all peripheral cycles of L in L' . This follows fairly easily from the description of the vertex sets of the halfspaces H_C^+ , as given in the proof of Lemma 2.3. As a consequence, we have

$$\partial_\infty \Sigma = \bigcap_C \bigcap_{\bar{w} \in W/W_C} \partial_\infty(\bar{w} \cdot H_C^+).$$

From this we get that

$$\partial_\infty \Sigma' \setminus \partial_\infty \Sigma = \partial_\infty \Sigma' \setminus \left(\bigcap_C \bigcap_{\bar{w} \in W/W_C} \partial_\infty(\bar{w} \cdot H_C^+) \right) = \bigcup_C \bigcup_{\bar{w} \in W/W_C} \left(\partial_\infty \Sigma' \setminus \partial_\infty(\bar{w} \cdot H_C^+) \right),$$

which clearly yields the assertion.

Denoting by $\text{Per}(L, L')$ the set of all peripheral cycles of L in L' , we can summarize the above discussion as follows.

2.6 Corollary. *Let L be an $(n+1)$ -sphere with holes distinct from a triangulation of the sphere S^n , and let $L < L' \cong S^{n+1}$ be an embedding as described at the beginning of this section. Then*

$$\mathcal{U} = \{\partial_\infty \Sigma' \setminus \partial_\infty(\bar{w} \cdot H_C^+) : C \in \text{Per}(L, L'), \bar{w} \in W/W_C\}$$

is the family of all connected components of the complement $\partial_\infty \Sigma' \setminus \partial_\infty \Sigma$. Each $U \in \mathcal{U}$ is an open $(n+1)$ -cell in $\partial_\infty \Sigma' = S^{n+1}$. If $U = \partial_\infty \Sigma' \setminus \partial_\infty(\bar{w} \cdot H_C^+)$ then its closure \bar{U} and its topological boundary $\text{bd}(U)$ in $\partial_\infty \Sigma'$ can be expressed as $\bar{U} = \partial_\infty(\bar{w} \cdot H_C^-)$ and $\text{bd}(U) = \partial_\infty(\bar{w} \cdot \Sigma_C)$.

Proof of Proposition 2.1(1): In the case when L is a triangulation of S^n , the assertion follows from Corollary 3.3 (given in the next section). In the remaining case, it follows from Corollary 2.6.

Proof of Proposition 2.1(2): Denote by $\text{vcd}(W)$ the virtual cohomological dimension of W . It follows from results of Mike Davis that

$$\text{vcd}(W) = \max\{k : \bar{H}^{k-1}(L \setminus \sigma) \neq 0, \text{ for some simplex } \sigma \text{ of } L, \text{ or } \bar{H}^{k-1}(L) \neq 0\}$$

(see Corollary 8.5.5 in [Da]). Moreover, by the fact that W acts geometrically on Σ and is virtually torsion-free, the pair $(\Sigma \cup \partial_\infty(W, S), \partial_\infty(W, S))$ is a \mathcal{Z} -structure (in the sense of Bestvina described in [Be]) for any torsion-free finite index subgroup $H < W$. Since, by Theorem 1.7 of [Be] we then have $\dim \partial_\infty(W, S) = \text{cd}(H) - 1$ (where $\text{cd}(H)$ is the cohomological dimension of H), it follows that $\dim \partial_\infty(W, S) = \text{vcd}(W) - 1$. Since L is a proper subpolyhedron in S^{n+1} , we get (from the above formula of Davis) that $\text{vcd}(W) \leq n$, and hence $\dim(\partial_\infty(W, S)) \leq n$.

On the other hand, being an $(n+1)$ -sphere with holes, L contains a full subcomplex C which is a PL triangulation of the sphere S^n . Denote by (W_C, S_C) the Coxeter system for the special subgroup $W_C < W$ induced by C . By Theorem (3b.2)(iii) of [DJ], the boundary $\partial_\infty(W_C, S_C)$ is then homeomorphic to S^n , and thus the boundary $\partial_\infty(W, S)$ contains an embedded copy of S^n . Hence $\dim(\partial_\infty(W, S)) \geq n$, which completes the proof.

3. Visual boundary of a $CAT(0)$ manifold with convex boundary.

It is known that the visual boundary of a complete $CAT(0)$ PL $(n+1)$ -manifold is homeomorphic to the sphere S^n (see [DJ], Theorem (3b.2)(iii)). The aim of this section is to extend this result as follows.

3.1 Proposition. *Let H be a complete piecewise euclidean $CAT(0)$ PL-manifold of dimension $n+1$ with nonempty connected convex boundary $N = \partial H$. Then the pair of visual boundaries $(\partial_\infty H, \partial_\infty N)$ is homeomorphic to the pair (B^n, S^{n-1}) , where B^n is the standard closed euclidean n -ball and $S^{n-1} = \partial B^n$.*

Before getting to the proof of the proposition, we present its two immediate corollaries, which are of independent interest.

3.2 Corollary. *Let M be a complete piecewise euclidean PL $(n+1)$ -manifold and N its closed connected PL submanifold of dimension n . Suppose that M is $CAT(0)$ and N is*

convex in M . Then the pair of visual boundaries $(\partial_\infty M, \partial_\infty N)$ is homeomorphic to the standard pair of spheres (S^n, S^{n-1}) .

Next corollary concerns arbitrary Coxeter groups, not only the right-angled ones.

3.3 Corollary. *Let (W', S') be a Coxeter system whose nerve L' is a PL triangulation of the n -sphere. Let L be a full subcomplex of L' such that the pair (L', L) is PL homeomorphic to the standard pair of spheres (S^n, S^{n-1}) . Let S be the vertex set of L , and denote by (W, S) the Coxeter system of the special subgroup induced by S . Then the pair of visual boundaries $(\partial_\infty(W', S'), \partial_\infty(W, S))$ is homeomorphic to the standard pair of spheres (S^n, S^{n-1}) .*

Proof: Denote by Σ' and Σ the Coxeter-Davis complexes of the systems (W', S') and (W, S) , respectively. Viewed naturally as a subcomplex, Σ is convex in Σ' . Moreover, under assumptions on L' and L , Σ' is a complete piecewise euclidean $CAT(0)$ PL-manifold of dimension $n + 1$, and Σ is its convex closed PL submanifold of dimension n . Corollary 3.2 yields then the assertion.

The rest of this section is devoted to the proof of Proposition 3.1. In this proof we follow the lines of the proof of Theorem (3b.2)(iii) in [DJ] (see also the appendix in [Sw1]). In particular, we use the technique of cellular sets and cell-like maps, whose basic elements we now recall.

In our setting below, a manifold is a (not necessarily compact) topological manifold with boundary (possibly empty). A compact subset E of a manifold M is *cellular* if the quotient space M/E (obtained by collapsing E to a point) is homeomorphic to M , and the quotient map $q : M \rightarrow M/E$ can be approximated by homeomorphisms. The latter means that for any fixed metric d in M/E (compatible with the topology) and any continuous positive function $\delta : M \rightarrow R_+$ there is a homeomorphism $h : M \rightarrow M/E$ such that for all $x \in M$ we have $d(q(x), h(x)) < \delta(x)$.

Cellular subsets of a manifold can be also characterised as admitting systems of neighbourhoods of a special form. To make a precise statement, recall that an *n -cell* in an n -dimensional manifold M is any subset $B \subset M$ which in some neighbourhood in M looks like:

- (1) B^n in R^n , if $B \subset \text{int}(M)$;
- (2) B_+^n in R_+^n , if $B \cap \partial M \neq \emptyset$, where $R_+^n = \{(x_1, \dots, x_n) : x_n \geq 0\}$ and $B_+^n = \{x \in R_+^n : |x| \leq 1\}$.

An *open n -cell* in M is a subset of an n -cell corresponding to the open ball $(B^n)^\circ = B^n \setminus \partial B^n$ or the open half-ball $(B_+^n)^\circ = \{x \in R_+^n : |x| < 1\}$, respectively.

3.4 Lemma (see Theorem 2.2 in [He]). *A compact subset E in an n -dimensional manifold M is cellular iff it admits arbitrarily close neighbourhoods in M which are open n -cells. More precisely, fixing any metric d in M (compatible with the topology), for any $\varepsilon > 0$ there is an open n -cell U in M such that $E \subset U \subset N_\varepsilon(E)$, where $N_\varepsilon(E) = \{x \in M : d(x, E) < \varepsilon\}$.*

A nonempty compact metric space of finite topological dimension is *cell-like* if it can be embedded as a cellular subset in some manifold (for justification of this characterization,

see the remark in the middle of page 114 in [Ed]). Obviously, each cellular subset of a manifold is cell-like, but the converse is not true (i.e. there are non-cellular embeddings of cell-like spaces in manifolds). A map between metric spaces is *cell-like* if it is a proper surjection and each point preimage is cell-like.

We now recall few preparatory results which will be needed in our proof of Proposition 3.1. The next result is due to Siebenmann [Si] for $n \geq 5$, Quinn [Qu] for $n = 4$, Armentrout [Ar] for $n = 3$ and R. L. Moore [Mo] for $n \leq 2$. The case of manifolds with nonempty boundary is carefully addressed in [Si], and the cases involving dimension 3 hold because the Poincare conjecture is true.

3.5 Approximation Theorem. *Each cell-like map between manifolds with boundary is a near-homeomorphism, i.e. it can be approximated by homeomorphisms. More precisely, if $f : M \rightarrow N$ is cell-like and d is any metric in N (compatible with the topology) then for any continuous positive function $\delta : M \rightarrow R_+$ there is a homeomorphism $h : M \rightarrow N$ such that for all $x \in M$ we have $d(f(x), h(x)) \leq \delta(x)$.*

The next result seems to be well known (it appears as Lemma 2 in the appendix to [Sw1], where its proof is included).

3.6 Lemma. *If $f : A \rightarrow B$ is a cell-like surjection between finite dimensional metric compacta, then A is cell-like if and only if B is cell-like.*

Next two preparatory results give some useful criteria for recognizing cellular subsets in spheres and disks. First of them presents a well known characterization of cellular subsets in spheres S^n as those which are *point-like* (i.e. such that the complement is homeomorphic to R^n). The second one is a less well known analogue for those subsets in disks which intersect the boundary.

3.7 Lemma (see Theorem on page 114 in [Ed]). *A closed subset E of the sphere S^n is cellular iff the complement $S^n \setminus E$ is homeomorphic to R^n .*

3.8 Lemma. *Let E be a closed subset of the n -disk D^n such that $E \cap \partial D^n \neq \emptyset$. Suppose that the complement $D^n \setminus E$ is homeomorphic to R_+^n . Then E is cellular in D^n .*

Proof: We need to indicate arbitrarily close open neighbourhoods of E in D^n which are open n -cells (of the form $(B_+^n)^\circ$). Let $h : R_+^n \rightarrow D^n \setminus E$ be a hypothesized homeomorphism, and let B_r denote the euclidean closed ball of radius r in R_+^n centered at 0. Obviously, as $r \rightarrow \infty$, the set $D^n \setminus h(B_r)$ becomes as close to E as necessary. We claim that $D^n \setminus h(B_r)$ is also an open n -cell in D^n . This follows fairly directly from the generalized Schoenflies theorem.

Having finished the preparations, we now state two auxilliary results which will lead us to the proof of Proposition 3.1. The term π -convexity appearing in Lemma 2 means that any geodesic of length less than π having both endpoints in the subspace, is entirely contained in this subspace.

Lemma 1. *Let n be a positive integer, and let H be a complete piecewise euclidean CAT(0) PL-manifold of dimension n , with nonempty connected convex boundary $N = \partial H$. Then for any $x \in N$ and any $R > 0$:*

- (B(n)) the sphere $S_R(x, H)$ is an $(n-1)$ -manifold and $S_R(x, N)$ is its boundary;
- (C(n)) for any $R' > R$ the geodesic projection towards x , $\rho_{R', R} : S_{R'}(x, H) \rightarrow S_R(x, H)$, is a cell-like map;
- (D(n)) the pair $(S_R(x, H), S_R(x, N))$ is homeomorphic to the pair (B^{n-1}, S^{n-2}) .

Lemma 2. *Let n be a positive integer, and let P be a complete piecewise spherical CAT(1) PL-manifold of dimension n , with nonempty π -convex boundary $Q = \partial P$. Then for any $v \in Q$:*

- (A_S(n)) for any $R \in (0, \pi]$ the open metric ball $B_R(v, P)$ is homeomorphic to the open n -dimensional half disk, and $B_R(v, Q)$ corresponds to its boundary;
- (C_S(n)) for any $R < R' < \pi$ the geodesic projection towards v , $\rho_{R', R} : S_{R'}(v, P) \rightarrow S_R(v, P)$, is a cell-like map.

We will prove all the assertions of Lemmas 1 and 2 by induction with respect to the dimension n . For $n = 1$ all of these statements are obviously true. The main inductive step will consist of the following implications:

- $A_S(n-1) \Rightarrow [B(n) \wedge C_S(n) \wedge C(n)]$;
- $[B(n) \wedge C(n)] \Rightarrow D(n)$;
- $C_S(n) \Rightarrow A_S(n)$.

We will also show (prior to proving the lemmas) the implication $(\forall n C(n) \wedge D(n)) \Rightarrow$ Proposition 3.1.

Proof of Proposition 3.1 (assuming $\forall n C(n) \wedge D(n)$):

Let H and N satisfy the assumptions of the proposition (in particular, $\dim H = n+1$), and let $x \in N \subset H$ be any point. Recall that we have

$$\partial_\infty H = \varprojlim [\{S_R(x, H) : R > 0\}, \{\rho_{R', R} : R' > R\}]$$

and

$$\partial_\infty N = \varprojlim [\{S_R(x, N) : R > 0\}, \{\rho_{R', R}|_{S_{R'}(x, N)} : R' > R\}],$$

where $\rho_{R', R}$ are the geodesic projections. Since by $C(n+1)$ these bonding maps $\rho_{R', R}$ are all cell-like, it follows from Approximation Theorem 3.5 that they are near-homeomorphisms. By a result of M. Brown (Theorem 4 in [Br]), if we fix an increasing sequence of radii $R_i \rightarrow \infty$, then we can also choose homeomorphisms $h_i : S_{R_i}(x, H) \rightarrow S_{R_{i-1}}(x, H)$ (appropriately close to the maps $\rho_{R_i, R_{i-1}}$) such that the visual boundary $\partial_\infty H$ is homeomorphic to the inverse limit

$$\varprojlim [\{S_{R_i}(x, H)\}, \{h_i\}].$$

By $D(n+1)$, this inverse limit is homeomorphic to B^n . Moreover, according to Brown's description of an explicit homeomorphism $F : \partial_\infty H \rightarrow \varprojlim [\{S_{R_i}(x, H)\}, \{h_i\}]$ as above (provided in the proof of Theorem 1 in [Br]), any point $z \in \partial_\infty H$ which is represented by a string (z_i) (of the inverse system $[\{S_{R_i}(x, H)\}, \{\rho_{R_i, R_{i-1}}\}]$) is mapped by F to a point y represented by a string (y_i) given by $y_i = \lim_{j>i} h_i \circ \dots \circ h_{j-1}(z_j)$. In particular, if $z \in \partial_\infty N$, we have that all $z_i \in S_{R_i}(x, N)$, and consequently all $y_i \in S_{R_i}(x, N)$ too, because

homeomorphisms preserve boundary. It follows that under F the subset $\partial_\infty N \subset \partial_\infty H$ is mapped to the subset

$$\varprojlim[\{S_{R_i}(x, N)\}, \{h_i|_{S_{R_i}(x, N)}\}] \cong S^{n-1} \subset B^n \cong \varprojlim[\{S_{R_i}(x, H)\}, \{h_i\}].$$

Since we know that $\partial_\infty N$ is homeomorphic to S^{n-1} (by the result from [DJ] mentioned at the beginning of this section), it follows that F maps $\partial_\infty N$ onto the above subset, and this completes the proof.

We now turn to the inductive proof of Lemmas 1 and 2, following the scheme outlined right after the statement of Lemma 2.

Proof of the implication $A_S(n-1) \Rightarrow B(n)$:

We refer to the argument as in the proof of the implication $(L_{n-1}) \Rightarrow (T_n)$ in [DJ], in a long paragraph beginning at the end of page 371 and ending in the middle of the next page. This argument, without any change, shows that $S_R(x, H) \setminus S_R(x, N)$ is an $(n-1)$ -manifold. Thus, to get $B(n)$, it is sufficient to show that any $y \in S_R(x, N)$ has an open neighbourhood in $S_R(x, H)$ homeomorphic to the $(n-1)$ -dimensional open half-disk $(B_+^{n-1})^\circ$. For the latter, it is obviously sufficient to show that any y as above has a closed neighbourhood Δ in the closed ball $\overline{B}_R(x, H)$ such that the pair $(\Delta, \Delta \cap \overline{B}_R(x, N))$ is homeomorphic to the pair (B_+^n, B_0^{n-1}) , with y corresponding to a point on the boundary ∂B_0^{n-1} .

So let $y \in S_R(x, N)$ be any point. Denote by $v \in \text{Lk}(y, H)$ the point induced by the geodesic $[y, x]$. The argument from [DJ] mentioned above, after applying to our setting, shows that for small $\varepsilon > 0$ the ball of radius ε in $\overline{B}_R(x, H)$ centered at y is homeomorphic to the quotient

$$\overline{B}_{\pi/2}(v, \text{Lk}(y, H)) \times [0, \varepsilon] / \overline{B}_{\pi/2}(v, \text{Lk}(y, H)) \times \{0\} \cup S_{\pi/2}(v, \text{Lk}(y, H)) \times [0, \varepsilon],$$

where the intersection of this ε -ball with $B_R(x, N)$ corresponds to the subset

$$\overline{B}_{\pi/2}(v, \text{Lk}(y, N)) \times [0, \varepsilon] / \overline{B}_{\pi/2}(v, \text{Lk}(y, N)) \times \{0\} \cup S_{\pi/2}(v, \text{Lk}(y, N)) \times [0, \varepsilon]$$

and where y corresponds to the point obtained from the collapsed set. Now, it follows from $A_S(n-1)$ that the pair of open metric balls $(B_{\pi/2}(v, \text{Lk}(y, H)), B_{\pi/2}(v, \text{Lk}(y, N)))$ is homeomorphic to the standard pair of open balls $((B_+^{n-1})^\circ, (B_0^{n-2})^\circ)$. This easily implies the last assertion of the previous paragraph, and hence completes the proof.

Proof of the implication $A_S(n-1) \Rightarrow [C_S(n) \wedge C(n)]$:

We present a proof of the implication $A_S(n-1) \Rightarrow C(n)$, and it will become clear that the same argument yields also the implication $A_S(n-1) \Rightarrow C_S(n)$.

Let $p \in S_R(x, H)$, and denote shortly by ρ the geodesic projection $\rho_{R', R} : S_{R'}(x, H) \rightarrow S_R(x, H)$. To prove that ρ is cell-like, we need to show that the preimage $\rho^{-1}(p)$ is a cell-like set. We will show this for the case when $p \in S_R(x, N)$, by an argument following that in the proof of Lemma (3b.1) and Theorem (3b.2) in [DJ] (see also the appendix to [Sw1]). The argument in the remaining case $p \in S_R(x, H) \setminus S_R(x, N)$ is analogous, follows more

directly from that in [DJ], builds upon Lemma 3.7 rather than Lemma 3.8, and we omit its details.

So suppose $p \in S_R(x, N)$. Our proof consists of two steps which correspond to the following two claims.

Claim 1. *Let $y \in N = \partial H$, and let v be any point from the sublink $\text{Lk}(y, N) \subset \text{Lk}(y, H)$ (where the latter pair of links is homeomorphic to the standard pair $(B^{n-1}, \partial B^{n-1})$). Then the complement in $L := \text{Lk}(y, H)$ of the open ball $B_\pi(v, L)$ is a cellular subset of L .*

To prove Claim 1, denote the complement $L \setminus B_\pi(v, L)$ by $\text{Sh}_L(v)$ and call it the *shadow* of v in L . Since N is a complete $CAT(0)$ manifold, it satisfies the geodesic extension property, and since it is convex in H , it follows that the shadow $\text{Sh}_L(v)$ has nonempty intersection with the sublink $\text{Lk}(y, N)$. Thus, $\text{Sh}_L(v)$ is a closed subset in the $(n-1)$ -disk L , and it intersects the boundary ∂L . On the other hand, by $A_S(n-1)$, the complement $L \setminus \text{Sh}_L(v) = B_\pi(v, L)$ is homeomorphic to the open $(n-1)$ -dimensional half-disk. It follows then from Lemma 3.8 that $\text{Sh}(v, L)$ is cellular in L , hence the claim.

Claim 2. *If $p \in S_R(x, N)$ then the preimage $\rho^{-1}(p)$ is cell-like.*

To prove Claim 2 we use an idea communicated to us by Mike Davis. Suppose *a contrario* that the preimage $\rho^{-1}(p)$ is not cell-like. Consider all geodesic rays in H started at x and passing through p , and let $[x, \tilde{p}]$ be the largest geodesic segment contained in all of those geodesic rays (it may happen that $\tilde{p} = p$). Consider the link $\text{Lk}(\tilde{p}, H)$ and the point $\tilde{v} \in \text{Lk}(\tilde{p}, H)$ corresponding to the geodesic segment $[x, \tilde{p}]$. Then the shadow of \tilde{v} in $\text{Lk}(\tilde{p}, H)$ is nontrivial. Put $\tilde{R} = d_H(x, \tilde{p})$. Since \tilde{p} has a metric cone neighbourhood in H (see e.g. Theorem 7.16 on page 104 in [BH]), there is $\tilde{R}' > \tilde{R}$ such that the preimage $\rho_{\tilde{R}', \tilde{R}}^{-1}(p)$ is homeomorphic to the shadow $\text{Sh}_{\text{Lk}(\tilde{p}, H)}(\tilde{v})$. In particular, it follows from Claim 1 that $\rho_{\tilde{R}', \tilde{R}}^{-1}(p)$ is homeomorphic to a cellular subset of $\text{Lk}(\tilde{p}, H)$, and hence it is cell-like. Since we have $\rho^{-1}(p) = \rho_{\tilde{R}', \tilde{R}}^{-1}[\rho_{\tilde{R}', \tilde{R}}^{-1}(p)]$, it follows from Lemma 3.6 that there is $p_1 \in \rho_{\tilde{R}', \tilde{R}}^{-1}(p)$ such that $\rho_{\tilde{R}', \tilde{R}}^{-1}(p_1)$ is not cell-like.

Iterating the argument as above, we get an infinite sequence of points p_n lying on a common finite geodesic segment started at x with the following property: denoting by $v_n \in \text{Lk}(p_n, H)$ the point corresponding to the segment $[x, p_n]$, we have that the shadow $\text{Sh}_{\text{Lk}(p_n, H)}(v_n)$ is nontrivial. On the other hand, it is not hard to realize that for any geodesic segment $[x, y]$ the set of points $p \in [x, y]$ with such nontrivial shadows is discrete, and hence finite (cf. Fact 4 in the appendix to [Sw1]). This contradiction concludes the proof of Claim 2, and hence also the proof of the implication $A_S(n-1) \Rightarrow C(n)$.

Proof of the implication $[B(n) \wedge C(n)] \Rightarrow D(n)$:

Note that, if $r > 0$ is sufficiently small, the pair $(S_r(x, H), S_r(x, N))$ is homeomorphic to the pair (B^{n-1}, S^{n-2}) by the fact that x has a metric cone neighbourhood in H . Since by $C(n)$, for any $R > r$ the map $\rho_{R,r} : S_R(x, H) \rightarrow S_r(x, H)$ is cell-like, the assertion of $D(n)$ follows from Approximation Theorem 3.5 combined with $B(n)$.

Proof of the implication $C_S(n) \Rightarrow A_S(n)$:

Note that for small R the assertion of $A_S(n)$ follows from the existence of a metric cone neighbourhood of v in P . Fix some $r > 0$ such that $B_r(v, P)$ is a cone neighbourhood. It remains to prove the assertion for $R \in (r, \pi]$. Given any R as above, put $c := \frac{r}{R}$, and

consider the map $f : B_R(v, P) \rightarrow B_r(v, P)$ defined as follows. For any $p \in B_R(v, P)$ put $r_p := d_P(v, p)$, and then put $f(p) := \rho_{r_p, cr_p}(p)$. Obviously, f is a continuous surjection between open balls (which are n -manifolds), and we claim that f is cell-like. Indeed, obviously we have $f^{-1}(v) = \{v\}$, and for any $q \in B_r(v, P)$, $q \neq v$, putting $r_q = d_P(v, q)$, the preimage $f^{-1}(q)$ is easily seen to be homeomorphic to the set $\rho_{r_q/c, r_q}^{-1}(q)$. Since $r_q/c < R \leq \pi$, it follows from $C_S(n)$ that the latter set is cell-like, and hence f is cell-like as well. Now, applying Approximation Theorem 3.5 we get a homeomorphism $h : B_R(v, P) \rightarrow B_r(v, P)$, which shows that $B_R(v, P)$ is homeomorphic to the open n -dimensional half-disk. Since obviously we have $\partial B_R(v, P) = B_R(v, Q)$, the second part of the assertion of $A_S(n)$ follows as well, which completes the proof.

4. 3-convexity in simplicial complexes.

In this section we present the concept of 3-convexity for subcomplexes in simplicial complexes, and discuss existence of various classes of examples of simplicial complexes L satisfying all the assumptions of Theorem 1.3.

3-convexity.

Given a simplicial complex K , we consider the combinatorial distance function dist_K between its vertices given by minimizing the length (i.e. the number of edges) of polygonal paths in the 1-skeleton $K^{(1)}$ of K connecting these vertices.

4.1 Definition (3-convexity). A subcomplex J in a simplicial complex K is *3-convex* if it is a full subcomplex and if for any two vertices of J staying at distance 2 in K any path of length 2 in $K^{(1)}$ connecting these vertices is entirely contained in J .

Next lemma presents a useful property of 3-convex subcomplexes. Given a simplex σ in a simplicial complex K , we denote by $\text{st}_K(\sigma)$, and call the *star of σ in K* , the subcomplex of K being the union of all simplices of K containing σ .

4.2 Lemma. *A full subcomplex J of a flag simplicial complex K is 3-convex if and only if for any vertex $v \in J$ the union $J \cup \text{st}_K(v)$ is a full subcomplex of K . Moreover, if J is 3-convex in K then for any simplex $\sigma \in J$ the union $J \cup \text{st}_K(\sigma)$ is a full subcomplex of K .*

Proof: Let $\sigma \in J$ and let v_1, \dots, v_m be any vertices of $J \cup \text{st}_K(\sigma)$ pairwise connected with edges in K . By flagness of K , there is a simplex spanned on these vertices, and we denote it τ . We need to show that $\tau \subset J \cup \text{st}_K(\sigma)$.

If $v_1, \dots, v_m \in J$, we get $\tau \subset J$ by fullness of J , and this clearly yields the assertion. We will show that otherwise all vertices v_1, \dots, v_m are contained in $\text{st}_K(\sigma)$. Let $v \in \{v_1, \dots, v_m\}$ be a vertex not contained in J , and suppose that $u \in \{v_1, \dots, v_m\}$ is a vertex not contained in $\text{st}_K(\sigma)$. Then $u \in J$, and there is $w \in \sigma$ such that $\text{dist}_K(u, w) = 2$. By 3-convexity of J , the geodesic path (u, v, w) is then contained in J , which contradicts the choice of v . Thus, $\{v_1, \dots, v_m\} \subset \text{st}_K(\sigma)$ in this case, and the assertion follows by the fact that star of any simplex in a flag simplicial complex is its full subcomplex.

To complete the proof, it remains to show that if for each vertex $v \in J$ the union $J \cup \text{st}_K(v)$ is full in K then J is 3-convex in K . Let v_1, v_2 be any vertices of J lying at distance 2 in K , and let (v_1, w, v_2) be any geodesic between them in $K^{(1)}$. If $w \notin J$ then we have $[w, v_2] \not\subset J \cup \text{st}_K(v_1)$, contradicting fullness of the latter union.

Constructing 3-convex subcomplexes.

In this subsection, given any polyhedron P and its subpolyhedron Q , we present a construction of a flag triangulation of P for which Q forms a 3-convex subcomplex. This construction allows to justify the existence of many simplicial complexes L satisfying the assumptions of Theorem 1.3, in arbitrary dimension n (see Example 4.6 below).

4.3 Definition (relative barycentric subdivision). Let K be a simplicial complex and J its subcomplex. The *barycentric subdivision of K relative to J* , denoted K'_J , is the subdivision of K obtained as follows. First, subdivide barycentrically each 1-simplex of K not contained in J , and then, recursively with respect to the dimension, subdivide each higher dimensional simplex not contained in J by coning its already subdivided boundary. Alternatively, take as the vertex set of K'_J all vertices of J and the barycenters $b(\sigma)$ of all simplices σ of K not contained in J ; simplices of K'_J are spanned precisely by the sets of form $\{v_1, \dots, v_q, b(\sigma_1), \dots, b(\sigma_p)\}$, where $q \geq 0$, $p \geq 0$, the vertices v_1, \dots, v_q span a simplex τ of J (empty when $q = 0$), $\sigma_1, \dots, \sigma_p$ are some simplices of K not contained in J , and $\tau \subset \sigma_1 \subset \dots \subset \sigma_p$.

Note that the subcomplex $J \subset K$, with its original simplicial structure induced from K , is also a subcomplex of K'_J . Next few lemmas collect various further properties of relative barycentric subdivision.

4.4 Lemma. *If a subcomplex J of a simplicial complex K is flag then the subdivision K'_J is also flag. In particular, if J is a full subcomplex of a flag complex K then the subdivision K'_J is also flag.*

Proof: Let A be a set of vertices of K'_J which are pairwise connected with edges. Suppose $A = \{u_1, \dots, u_k, b(\rho_1), \dots, b(\rho_m)\}$, where u_1, \dots, u_k are some vertices of J and ρ_1, \dots, ρ_m are some simplices of K not contained in J . By the assumption on A , the vertices u_1, \dots, u_k are pairwise connected with edges of J , and since J is flag, they span a simplex τ in J . Again, by the assumption on A , for any two simplices ρ_i, ρ_j one of them is a face of the other. Consequently, we may assume that $\rho_1 \subset \dots \subset \rho_m$. Moreover, the existence of edges in K'_J between $b(\rho_1)$ and any of the u_i implies that $\tau \subset \rho_1$. It follows then directly from the description of simplices in K'_J (in Definition 4.3) that the set A spans a simplex of K'_J . Hence flagness of K'_J .

4.5 Lemma. *If a subcomplex J of a simplicial complex K is flag and full in K then J is a 3-convex subcomplex of K'_J . In particular, if J is a full subcomplex of a flag complex K then J is 3-convex in K'_J .*

Proof: The fact that J is a full subcomplex of K'_J is obvious. To show that J is also 3-convex, consider any vertices v, w of J at combinatorial distance 2 in K'_J . Suppose on the contrary that there is a combinatorial geodesic (v, u, w) in the 1-skeleton of K'_J such that $u \notin J$. By definition of K'_J , the 1-simplex (v, u) must have the form (v, b_σ) for some simplex σ of K containing v , and (u, w) must have the form (b_τ, w) for some simplex τ of K containing w . It follows that $\sigma = \tau$, and we get that u and w belong to the same simplex of K . Since J is full in K , the edge (v, w) is an edge of J , and hence also an edge of K'_J , contradicting the fact that $\text{dist}_{K'_J}(v, w) = 2$.

4.6 Example. Using Lemmas 4.4 and 4.5, one can construct many simplicial complexes L satisfying the assumptions of Theorem 1.3, in arbitrary dimension n . Indeed, consider any PL triangulation N of the $(n+1)$ -sphere, and any collection $\sigma_1, \dots, \sigma_k$ of pairwise nonintersecting $(n+1)$ -simplices of N , with $k \geq 1$. Let M be the simplicial complex obtained from N by deleting interiors of all simplices $\sigma_1, \dots, \sigma_k$. Obviously, M is a PL triangulation of an $(n+1)$ -manifold with boundary (k times punctured $(n+1)$ -sphere). Let M' be the first barycentric subdivision of M , and let J be its subcomplex corresponding to the boundary. Then obviously M' is flag, and J is a full subcomplex of M' . Putting $L = (M')'_J$, we obtain (in view of Lemma 4.4) a flag simplicial complex for which J is (by Lemma 4.5) a 3-convex subcomplex. Consequently, the connected components of J (which correspond to subcomplexes $\text{bd}(\Omega)$ of Definition 1.2) are also 3-convex in L . This shows that L satisfies all the assumptions of Theorem 1.3.

Similarly, given any polyhedron P and its any subpolyhedron Q , and starting from any triangulation M of P for which Q induces a subcomplex, we can apply the same procedure as above to get a flag triangulation L of P for which Q induces a 3-convex subcomplex.

Flag-no-square spheres with holes having all peripheral cycles 3-convex.

In this subsection we discuss existence of examples of $(n+1)$ -spheres with holes, in various dimensions n , satisfying all the assumptions of Theorem 1.3, and such that the associated Coxeter group W is word-hyperbolic.

Recall that, given an $(n+1)$ -sphere with holes L (viewed as PL embedded in the sphere S^{n+1}), a *peripheral cycle* in L is any subcomplex C which is induced by the topological boundary $\text{bd}(\Omega)$ of a connected component Ω of the complement $S^{n+1} \setminus L$.

A simplicial complex K is said to satisfy the *flag-no-square* condition if it is flag and the following no-empty-square property holds: for any closed cycle γ of length 4 in the 1-skeleton of K there is at least one diagonal of γ (i.e. an edge connecting some pair of opposite vertices of γ) in K . Recall that a right-angled Coxeter group W is word-hyperbolic if and only if its nerve L satisfies the flag-no-square condition (see Corollary 12.6.3 in [Da], as applied to the right angled case).

We will first show an example of a 3-sphere with holes which is flag-no-square and which satisfies all the assumptions of Theorem 1.3. More precisely, by Lemma 4.8 below, the 580-disk described in Definition 4.7 is such an example. In view of Theorem 1.3, this leads to an example of a right-angled hyperbolic Coxeter group whose Gromov boundary is homeomorphic to the 2-dimensional Sierpiński compactum. On the other hand, we will show that for $n \geq 3$ there are no examples of $(n+1)$ -spheres with holes which satisfy the flag-no-square condition, and to which Theorem 1.3 applies (see Lemma 4.9 below).

Consider the 600-cell, the convex regular 4-polytope with Schläfli symbol $\{3, 3, 5\}$ (see e.g. [C]). Denote by X_{600} the boundary of the 600-cell, a 3-dimensional simplicial polyhedron homeomorphic to the 3-dimensional sphere. Recall that, when viewed as a simplicial complex, X_{600} consists of 600 3-simplices and has 120 vertices. Its vertex links are icosahedra and edge links are pentagons. We will only exploit the combinatorial simplicial structure of X_{600} .

4.7 Definition. The *580-disk* is the simplicial complex obtained from X_{600} by deleting open star of any vertex. We denote it by X_{580} .

Note that X_{580} is a PL triangulation of the 3-dimensional disk B^3 consisting of 580 3-simplices, and its boundary ∂X_{580} , which corresponds to the link of the deleted vertex, is an icosahedral triangulation of the 2-sphere.

4.8 Lemma. *The 580-disk X_{580} is a flag-no-square simplicial complex, and the boundary ∂X_{580} is its 3-convex subcomplex.*

Proof: To prove the lemma, we need the following fact, whose proof can be found in [PS] (see Lemma 2.1 in that paper).

Lemma 1. *X_{600} satisfies the flag-no-square property.*

Next two claims exhibit basic properties of the 580-disk, resulting from Lemma 1.

Claim 1. *The 580-disk X_{580} satisfies the flag-no-square property.*

To get Claim 1, note that X_{580} is a full subcomplex in X_{600} . This follows from a more general easy observation that, given any simplicial complex K and its any vertex v , the subcomplex of K obtained by deleting the open star of v is full. Since the flag-no-square property is inherited by full subcomplexes, the claim follows by referring to Lemma 1.

Claim 2. *The boundary ∂X_{580} is a full subcomplex of the 580-disk X_{580} .*

To get Claim 2, denote by D the closed star of this vertex of X_{600} which does not belong to X_{580} . Since closed star of any vertex in a flag simplicial complex is a full subcomplex, and since X_{600} is flag (see Lemma 1), we get that D is a full subcomplex of X_{600} . Since $\partial X_{580} = D \cap X_{580}$, the claim is a consequence of the following general fact: if A is a full subcomplex of a simplicial complex K then for any subcomplex B of K the intersection $A \cap B$ is a full subcomplex of B .

In view of Claims 1 and 2, it remains to show that, given any two vertices x, z of ∂X_{580} at combinatorial distance 2 in the 1-skeleton of X_{580} , and any polygonal path (x, y, z) connecting them in X_{580} , we have $y \in \partial X_{580}$. View X_{580} as a subcomplex in X_{600} , and let v be the vertex of X_{600} not contained in X_{580} . Consider the 4-cycle (x, y, z, v, x) in X_{600} . Since the combinatorial distance between x and z in the 1-skeleton of X_{600} is easily seen to be 2 (e.g. since X_{580} is full in X_{600}), it follows from Lemma 1 that y is adjacent to v in X_{600} . But this means that $y \in \partial X_{580}$, which completes the proof of Lemma 4.8.

Our final observation is the following lemma, which implies that using Theorem 1.3, one cannot construct an example of a right-angled word-hyperbolic Coxeter group whose Gromov boundary is an n -dimensional Sierpiński compactum, for any $n \geq 3$.

4.9 Lemma. *If $n \geq 3$, then there is no $(n + 1)$ -sphere with holes L satisfying conditions (1) and (2) of Theorem 1.3 and the flag-no-square condition.*

Proof: Let L be an $(n + 1)$ -sphere with holes satisfying the assumptions (1) and (2) of Theorem 1.3 and the flag-no-square condition. For any peripheral cycle C of L , attach to L the simplicial cone over C , thus getting a PL triangulation L' of the $(n + 1)$ -sphere. We will show that L' satisfies the flag-no-square condition.

Flagness of L' easily follows from flagness of L and the fact that the peripheral cycles are full subcomplexes of L . It remains to show that L' satisfies the no-empty-square

condition. More precisely, let $\gamma = (w, x, y, z, w)$ be a 4-cycle in the 1-skeleton of L' . We need to show that γ has at least one of the diagonals in L' .

If γ is contained in L , existence of a diagonal follows from the assumption that L is flag-no-square. Otherwise, γ has a vertex, say w , which is the cone vertex of one of the cones, say the cone over a peripheral cycle C , attached to L . We then have that x and z necessarily belong to C . If y is also a vertex of C , the edge (w, y) is a diagonal of γ . If x is adjacent to z in C , the edge (x, z) is a diagonal of γ . It remains to consider the last case when x is not adjacent to z in C and y is not contained in C .

The last case mentioned above has two subcases. In the first subcase, suppose that y is a vertex of L . Then existence of the path (x, y, z) contradicts 3-convexity of C in L , and hence this subcase is not possible. In the second and last subcase, y has to be the cone vertex of the cone over a peripheral cycle C' of L distinct from C . But then the non-adjacent vertices x, z belong to the intersection $C \cap C'$, which contradicts the assumption that such intersection is either empty or a single simplex of L . So this subcase is also not possible, and hence L' satisfies the flag-no-square condition.

The lemma is now a consequence of nonexistence of flag-no-square triangulations of spheres S^{n+1} for $n \geq 3$ (cf. Corollary 5.7 in [PS]).

Lemma 4.9, and the comment right before its statement, suggest the following.

4.10 Question. Given an integer $n \geq 3$, does there exist a right-angled Coxeter group W which is word-hyperbolic and whose Gromov boundary ∂W is an n -dimensional Sierpiński compactum?

5. Local 3-convexity in CAT(0) cubical complexes.

In this section we use the concept of 3-convexity to present some strengthened notion of convexity (called *local 3-convexity*) in CAT(0) cubical complexes. We then derive, as Proposition 5.3 below, some rather strong global geometric property of locally 3-convex subcomplexes. This property is essential for our arguments in Section 6.

5.1 Definition (local 3-convexity). A subcomplex Y of a cubical complex X is *locally 3-convex* if for each vertex $v \in Y$ the link $\text{Lk}(v, Y)$ is a 3-convex subcomplex in the corresponding link $\text{Lk}(v, X)$.

Before stating the main result of this section, concerning some global behaviour of locally 3-convex subcomplexes, we make few preparations.

To each complete convex subspace Y of a CAT(0) space X (in particular, to any convex subcomplex Y of a CAT(0) cubical complex X) there is associated a map $\pi : X \rightarrow Y$, called *projection to Y* , given by $\pi(x) = y$, where y is the unique point of Y closest to x (cf. Chapter II.2 in [BH]). The following observation will be used later in our arguments.

5.2 Lemma. *Let X be a CAT(0) cubical complex and Y its convex subcomplex. Then for any vertex v of X its projection to Y is also a vertex.*

Proof: Suppose that $\pi(x) = y$, where $y \in Y$ is not a vertex. We need to show that x is not a vertex. Let σ be the cubical face of Y such that y belongs to the interior of σ . Let $[x, y]$ denote the geodesic in X from x to y . Since y is the point of Y closest to x , there is a face

τ in X containing σ and such that the final part of $[x, y]$ is contained in τ and orthogonal to σ . In particular, there is a pair of opposite codimension 1 faces in τ such that the part of $[x, y]$ contained in τ is parallel to those two faces, and disjoint with them both. Then, using induction, we observe that $[x, y]$ satisfies the same property with respect to any cubical face τ' of X that it crosses (here we use the fact that $[x, y]$ is the concatenation of finitely many segments, each of which is contained in a single cubical face, cf. Corollary 7.29 on p. 110 in [BH]). In particular, it follows that $[x, y]$ omits all vertices of any cubical face of X that it crosses, and hence x is not a vertex of X . This completes the proof.

We denote by $\text{st}_c(v, X)$, and call the *cubical star* of a vertex v in a CAT(0) cubical complex X , the subcomplex of X equal to the union of all cubical faces of X containing v .

5.3 Proposition. *Let X be a CAT(0) cubical complex, and let Y be its convex subcomplex which is locally 3-convex. Let v be any vertex of X , and denote by u the vertex of Y equal to the projection of v to Y . Let ρ be any geodesic ray in X started at v and representing a point $\xi \in \partial_\infty Y \subset \partial_\infty X$. Then*

- (a) ρ intersects the subcomplex $\text{st}_c(u, Y)$;
- (b) except for a bounded initial subsegment, ρ is contained in Y .

Proof: The proof is split into several parts, and it occupies the rest of this section.

Thickened hyperplanes and extended half-spaces.

We refer to hyperplanes and half-spaces in CAT(0) cubical complexes, as described e.g. in [W], Sections 2.4, 3.2 and 3.3. Given any oriented edge $e = (p, q)$ of X , denote by J_e the hyperplane in X dual to e , i.e. the hyperplane orthogonal to e and passing through the midpoint of e . Denote by H_e this closed half-space for J_e which contains p . Recall that both J_e and H_e are convex subspaces in X , and that the hyperplane J_e is canonically a CAT(0) cubical complex.

Denote by J_e^* , and call the *thickened hyperplane* in X dual to e , the subcomplex of X being the union of all cubical faces of X intersected by J_e . Note that, as a cubical complex, J_e^* is isomorphic to the product $J_e \times I$, where I is the 1-cube. Moreover, given any vertex o in J_e^* , there is a unique edge ε in X containing o and intersecting J_e ; denoting by t_ε the vertex in the link $\text{Lk}(o, X)$ induced by ε , we have $\text{Lk}(o, J_e^*) = \text{st}(t_\varepsilon, \text{Lk}(o, X))$. In particular, $\text{Lk}(o, J_e^*)$ is a full subcomplex of $\text{Lk}(o, X)$, and hence any thickened hyperplane J_e^* is a convex subcomplex of X .

Denote by H_e^* , and call the *extended half-space* of X induced by e , the subcomplex equal (as a set) to the union $H_e \cup J_e^*$. Since for any vertex $s \in H_e^*$ either $\text{Lk}(s, H_e^*) = \text{Lk}(s, X)$ or $\text{Lk}(s, H_e^*) = \text{Lk}(s, J_e^*)$, we get that any extended half-space H_e^* is a convex subcomplex of X .

Compare [W], Section 3.3, where extended hyperplanes are called hyperplane carriers, and extended half-spaces are called halfspace carriers.

The subcomplex P and its basic properties.

Given X, Y, v and u as in the proposition, denote by \mathcal{E}_u^Y the set of all oriented edges in Y with the initial vertex u , and put

$$P := \bigcap_{e \in \mathcal{E}_u^Y} H_e^*.$$

Obviously, P is a convex subcomplex of X .

Claim 1. $P \cap Y = \text{st}_c(u, Y)$.

To prove Claim 1, note that for each $e \in \mathcal{E}_u^Y$, if we denote by $J_{e,Y}$ the hyperplane in Y dual to e , we have $J_{e,Y} = J_e \cap Y$. Similarly, if $H_{e,Y}^*$ denotes the extended half-space in Y induced by e , we have $H_{e,Y}^* = H_e^* \cap Y$. Consequently, we have

$$P \cap Y = \left(\bigcap_{e \in \mathcal{E}_u^Y} H_e^* \right) \cap Y = \bigcap_{e \in \mathcal{E}_u^Y} (H_e \cap Y) = \bigcap_{e \in \mathcal{E}_u^Y} H_{e,Y}^*,$$

and the latter intersection obviously coincides with the cubical star $\text{st}_c(u, Y)$, thus justifying the claim.

Claim 2. $v \in P$.

Since $H_e \subset H_e^*$, to prove Claim 2, it is sufficient to show that for each $e \in \mathcal{E}_u^Y$ we have $v \in H_e$. To do this, suppose it is not true, i.e. $v \notin H_e$ for some $e \in \mathcal{E}_u^Y$. Since $u \in H_e$, the geodesic $[v, u]$ crosses the hyperplane J_e . Let z be the point in the intersection $[v, u] \cap J_e$ closest to u . Consider the geodesic triangle $T = \Delta(u, m, z)$, where m is the midpoint of the edge e . Since $z \in J_e$, the angle of T at m equals $\pi/2$. Since the side $[u, m]$ of T is contained in Y , and since u is the projection of z to Y , the angle of T at u is $\geq \pi/2$. Finally, the angle of T at z is > 0 . This contradicts the fact that the sum of angles in any geodesic triangle in a CAT(0) space is $\leq \pi$, thus completing the proof of Claim 2.

Convexity of $Y \cup P$.

To prove that the subcomplex $Y \cup P$ is convex, observe first that it is obviously connected. It is then sufficient to show that for any vertex $z \in Y \cup P$ the link $\text{Lk}(z, Y \cup P)$ is a full subcomplex in the link $\text{Lk}(z, X)$. Obviously, this is true for any vertex $z \in Y \cup P$ disjoint with the intersection $P \cap Y$, by convexity of both P and Y . It remains to check this property for vertices $z \in P \cap Y$, i.e. for vertices $z \in \text{st}_c(u, Y)$, see Claim 1 above.

For $z = u$ we have $\text{Lk}(u, P \cup Y) = \text{Lk}(u, X)$, because $\text{st}_c(u, X) \subset P \cup Y$, and hence the above mentioned property follows trivially. Suppose $z \in P \cap Y = \text{st}_c(u, Y)$, and $z \neq u$. Let σ be the smallest cubical face of X containing u and z . Let e_1, \dots, e_m be the edges adjacent to u in σ , oriented so that u is their initial vertex. Obviously, we have $e_1, \dots, e_m \in \mathcal{E}_u^Y$. Let d_1, \dots, d_m be the edges of σ adjacent to z and parallel to e_1, \dots, e_m respectively. Equip them with the orientations consistent with those of e_1, \dots, e_m , i.e. such that z is their terminal vertex. Denote by t_1, \dots, t_m the vertices in $\text{Lk}(z, X)$ induced by the edges d_1, \dots, d_m respectively. Denote also by τ the simplex of $\text{Lk}(z, X)$ spanned by the vertices t_1, \dots, t_m (i.e. induced by the face σ). We then have

$$\text{Lk}(z, P) = \text{Lk}\left(z, \bigcap_{e \in \mathcal{E}_u^Y} H_e^*\right) = \bigcap_{e \in \mathcal{E}_u^Y} \text{Lk}(z, H_e^*) = \text{Lk}(z, H_{e_1}^*) \cap \dots \cap \text{Lk}(z, H_{e_m}^*),$$

where the last equality follows by an easy observation that for $e \in \mathcal{E}_u^Y \setminus \{e_1, \dots, e_m\}$ we have $\text{Lk}(z, H_e^*) = \text{Lk}(z, X)$. As a consequence, we get

$$\text{Lk}(z, P \cup Y) = \text{Lk}(z, Y) \cup \text{Lk}(z, P) = \text{Lk}(z, Y) \cup [\text{Lk}(z, H_{e_1}^*) \cap \dots \cap \text{Lk}(z, H_{e_m}^*)] =$$

$$\begin{aligned}
&= \text{Lk}(z, Y) \cup [\text{Lk}(z, H_{d_1}^*) \cap \dots \cap \text{Lk}(z, H_{d_m}^*)] = \\
&= \text{Lk}(z, Y) \cup [\text{Lk}(z, J_{d_1}^*) \cap \dots \cap \text{Lk}(z, J_{d_m}^*)] = \\
&= \text{Lk}(z, Y) \cup [\text{st}(t_1, \text{Lk}(z, X)) \cap \dots \cap \text{st}(t_m, \text{Lk}(z, X))] = \text{Lk}(z, Y) \cup \text{st}(\tau, \text{Lk}(z, X)).
\end{aligned}$$

Since $\text{Lk}(z, Y)$ is 3-convex in $\text{Lk}(z, X)$, and since $\tau \subset \text{Lk}(z, Y)$, it follows from the above equality and from the second assertion of Lemma 4.2 that $\text{Lk}(z, P \cup Y)$ is a full subcomplex of $\text{Lk}(z, X)$. This completes the proof of convexity of the subcomplex $P \cup Y \subset X$.

Proof of assertions (a) and (b) of the proposition.

Let ρ be a geodesic ray as in the proposition. If $v \in Y$, assertion (a) is obvious since $u = v$, and assertion (b) follows from convexity of Y . So assume that $v \notin Y$. By the fact that $Y \cup P$ is convex, it makes sense to speak of the subset $\partial_\infty(Y \cup P) \subset \partial_\infty X$. Since $\partial_\infty Y \subset \partial_\infty(Y \cup P)$, we have $\xi \in \partial_\infty(Y \cup P)$, and since $v \in P \subset Y \cup P$, we get that ρ is contained in $Y \cup P$. To prove (a), suppose on the contrary that ρ does not intersect $\text{st}_c(u, Y) = P \cap Y$. Since $P \setminus (P \cap Y)$ is open and closed in $(P \cup Y) \setminus (P \cap Y)$, and since $v \in P \setminus (P \cap Y)$, the ray ρ must be contained in $P \setminus (P \cap Y)$. Since $P \cap Y$ is bounded, it follows that $d_{Y \cup P}(\rho(t), Y) = d_X(\rho(t), Y \cap P) \rightarrow \infty$ as $t \rightarrow \infty$. By convexity of $Y \cup P$, the distances in $Y \cup P$ are the same as those in X , and hence $d_X(\rho(t), Y) \rightarrow \infty$ as $t \rightarrow \infty$, which contradicts the assumption that $\lim_{t \rightarrow \infty} \rho(t) = \xi \in \partial_\infty Y$. Thus assertion (a) follows. A similar argument shows that, after leaving $P \cap Y$, the geodesic ray ρ stays in Y , which yields assertion (b).

6. Completing the proof of Theorem 1.3.

To prove Theorem 1.3, we consider (under notation and assumptions of this theorem) an embedding of the visual boundary $\partial_\infty(W, S)$ in the sphere S^{n+1} as described in Section 2. We need to verify that, under this embedding, $\partial_\infty(W, S)$ satisfies conditions (1)–(4) of Definition 1.1. Note that conditions (1) and (2) have been already verified in Section 2 (see Proposition 2.1(2) and Corollary 2.6). It remains to verify conditions (3) and (4), which we do below, in Propositions 6.1 and 6.4.

Closures of complementary components are pairwise disjoint.

This subsection is devoted to the proof of the following.

6.1 Proposition. *Let (W, S) be a right-angled Coxeter system whose nerve L is an $(n+1)$ -sphere with holes, and suppose that the following two conditions hold:*

- (1) *for any two distinct connected components of $S^{n+1} \setminus L$, their closures in S^{n+1} are either disjoint, or intersect at a single simplex of L ;*
 - (2) *the boundary $\text{bd}(\Omega)$ of any component Ω of $S^{n+1} \setminus L$ is a 3-convex subcomplex of L .*
- Consider an embedding of the boundary $\partial_\infty(S, W)$ in S^{n+1} , as described in Section 2. Then the closures of any two distinct connected components of $S^{n+1} \setminus \partial_\infty(W, S)$ are disjoint.*

We use the setting and the notation fixed at the beginning of Section 2. In particular, $\partial_\infty(W, S) = \partial_\infty \Sigma$ is realized as a subspace of $S^{n+1} = \partial_\infty \Sigma' = \partial_\infty(W', S')$, for appropriate choice of the nerve $L' \supset L$. By Lemma 2.5, connected components of $S^{n+1} \setminus \partial_\infty(W, S) =$

$\partial_\infty \Sigma' \setminus \partial_\infty \Sigma$ are parametrized by the pairs $(C, \bar{w}) : C \in \text{Per}(L, L'), \bar{w} \in W/W_C$ and have form

$$U = U_{C, \bar{w}} = \partial_\infty \Sigma' \setminus \partial_\infty (\bar{w} \cdot H_C^+) = \partial_\infty (\bar{w} \cdot H_C^-) \setminus \partial_\infty (\bar{w} \cdot \Sigma_C).$$

The topological boundary $\text{bd}(U)$ coincides then with the set $\partial_\infty (\bar{w} \cdot \Sigma_C)$. Thus, to prove Proposition 6.1, it is sufficient to show the following.

6.2 Lemma. *Under assumptions of Proposition 6.1, if $(C_1, \bar{w}_1), (C_2, \bar{w}_2)$ are distinct pairs such that $C_i \in \text{Per}(L, L')$ and $\bar{w}_i \in W/W_{C_i}$ then the sets $\partial_\infty (\bar{w}_1 \cdot \Sigma_{C_1})$ and $\partial_\infty (\bar{w}_2 \cdot \Sigma_{C_2})$ are disjoint (as subsets of $\partial_\infty \Sigma$).*

Proof: We will need the following.

Claim. *If the intersection $\bar{w}_1 \cdot \Sigma_{C_1} \cap \bar{w}_2 \cdot \Sigma_{C_2}$ is not empty then it consists of a single cubical cell of the complex Σ .*

To prove the claim, we refer here to the combinatorial description of the Coxeter-Davis complex Σ , as a cubical complex, given in [Da], Chapter 7, Section 7.3. In this description, the cubical cells of Σ are in 1-1 correspondence with the cosets of finite special subgroups of W , and in particular the vertices of Σ are in 1-1 correspondence with W . For any $C \in \text{Per}(L, L')$ and any $\bar{w} = w \cdot W_C \in W/W_C$, the vertex set of the subcomplex $\bar{w} \cdot \Sigma_C$ can be identified with the subset $w \cdot W_C \subset W$. Obviously, if $C_1 = C_2 = C$ and $\bar{w}_1 = w_1 \cdot W_C \neq \bar{w}_2 = w_2 \cdot W_C$ then the subsets $w_1 \cdot W_{C_1}$ and $w_2 \cdot W_{C_2}$ are disjoint. It follows that under assumption of the claim we have $C_1 \neq C_2$. Now, since $\bar{w}_1 \cdot \Sigma_{C_1} \cap \bar{w}_2 \cdot \Sigma_{C_2} \neq \emptyset$, there is a vertex $w \in W$ which belongs to this intersection. We then have $\bar{w}_1 \cdot \Sigma_{C_1} = w \cdot \Sigma_{C_1}$ and $\bar{w}_2 \cdot \Sigma_{C_2} = w \cdot \Sigma_{C_2}$. Consequently, by invariance under W , to prove the claim it is sufficient to show that the intersection $\Sigma_{C_1} \cap \Sigma_{C_2}$ is a single cell. The vertex sets of the subcomplexes Σ_{C_1} and Σ_{C_2} correspond to the subgroups $W_{C_1}, W_{C_2} < W$, respectively. Obviously, we have $W_{C_1} \cap W_{C_2} = W_{C_1 \cap C_2}$, where W_\emptyset denotes the trivial subgroup. By the assumption (1) of Proposition 6.1, $C_1 \cap C_2$ is either empty, or a single simplex of L . The special subgroup $W_{C_1 \cap C_2}$ is then finite. As a consequence, the intersection $\Sigma_{C_1} \cap \Sigma_{C_2}$ is exactly the cell of Σ corresponding to the subgroup $W_{C_1 \cap C_2}$ viewed as a coset of a finite special subgroup in W , hence the claim.

To conclude the proof of Lemma 6.2, consider any two points ξ_1, ξ_2 belonging to the sets $\partial_\infty (\bar{w}_1 \cdot \Sigma_{C_1})$ and $\partial_\infty (\bar{w}_2 \cdot \Sigma_{C_2})$, respectively. Fix any vertex v of Σ , and denote by ρ_1, ρ_2 the geodesic rays in Σ started at v and representing the points ξ_1, ξ_2 respectively. Note that for any vertex u of any subcomplex $\bar{w}_i \cdot \Sigma_{C_i}$ the pair of vertex links $(\text{Lk}(u, \Sigma), \text{Lk}(u, \bar{w}_i \cdot \Sigma_{C_i}))$ is isomorphic (as a pair of simplicial complexes) to the pair (L, C_i) . By assumption (2) of Proposition 6.1, it follows that each $\bar{w}_i \cdot \Sigma_{C_i}$ is a convex and locally 3-convex subcomplex of Σ . Applying Proposition 5.3(b), we get that each of the rays ρ_i , except for a finite subsegment, is contained in the corresponding subcomplex $\bar{w}_i \cdot \Sigma_{C_i}$. In view of the claim, this implies that $\rho_1 \neq \rho_2$, and consequently $\xi_1 \neq \xi_2$, hence the lemma.

By what was said above, this also concludes the proof of Proposition 6.1.

Natural metrics on the visual boundary of a CAT(0) space.

In this subsection we describe a family of natural metrics (compatible with the cone topology) on the visual boundary $\partial_\infty X$ of a complete CAT(0) space X . (A metric of

this form will be used in the argument in the next subsection.) Recall that in a complete CAT(0) space any point $x_0 \in X$ can be connected to any $\xi \in \partial_\infty X$ with the unique geodesic ray (see [BH], Chapter II, Proposition 8.19). Recall also that for distinct geodesic rays r, r' based at a common point of X the function $t \rightarrow d_X(r(t), r'(t))$ is convex (see [BH], Chapter II, Proposition 2.2), and hence for any real $A > 0$ there is exactly one t with $d_X(r(t), r'(t)) = A$. The following fact was observed by Damian Osajda, see Proposition 9.6(1) in [OS].

6.3 Lemma. *Let X be a complete CAT(0) space, $x_0 \in X$ a point, and $A > 0$ a real number. For $\xi, \eta \in \partial_\infty X, \xi \neq \eta$, let r_ξ, r_η be the unique geodesic rays from x_0 to ξ and η , respectively. Put $d_A(\xi, \eta) := t^{-1}$, where t is the unique number with $d_X(r_\xi(t), r_\eta(t)) = A$. Then d_A is a metric on $\partial_\infty X$ compatible with the cone topology.*

The family of complementary components is null.

The last missing element in the proof of Theorem 1.3 is the following.

6.4 Proposition. *Let (W, S) be a right-angled Coxeter system whose nerve L is an $(n+1)$ -sphere with holes, and suppose that the following two conditions hold:*

- (1) *for any two distinct connected components of $S^{n+1} \setminus L$, their closures in S^{n+1} are either disjoint, or intersect at a single simplex of L ;*
 - (2) *the boundary $\text{bd}(\Omega)$ of any component Ω of $S^{n+1} \setminus L$ is a 3-convex subcomplex of L .*
- Consider an embedding of the boundary $\partial_\infty(S, W)$ in S^{n+1} , as described in Section 2. Then the family of connected components of $S^{n+1} \setminus \partial_\infty(W, S)$ is null.*

Proof: We come back to the notation introduced at the beginning of Section 2. Note that, in view of Corollary 2.6, the family of closures of the connected components of $S^{n+1} \setminus \partial_\infty(W, S)$ coincides with the family

$$\partial_\infty(\bar{w} \cdot H_C^-) : C \in \text{Per}(L, L'), \bar{w} \in W/W_C.$$

Obviously, it is then sufficient to show that this last family is null.

Order the family of all half-spaces in Σ' of form $\bar{w} \cdot H_C^-$ as above into a sequence $(H_k)_{k \geq 1}$, and recall that these half-spaces are all convex. Fix any vertex $v_0 \in \Sigma'$, and for each $k \geq 1$ denote by u_k the projection of v_0 to H_k (which is a vertex, by Lemma 5.2).

Claim 1. *Put $t_k = d_{\Sigma'}(v_0, u_k)$. Then $\lim_{k \rightarrow \infty} t_k = \infty$.*

To prove Claim 1, note that each half-space H_k is the union of a collection of $(n+2)$ -cubes of Σ' , and that two different half-spaces H_k have no such $(n+2)$ -cube in common. The claim follows then from uniform finiteness of Σ' .

Claim 2. *Each half-space H_k is a convex and locally 3-convex subcomplex of Σ' .*

To prove Claim 2, we only need to check that H_k is locally 3-convex in Σ' . Assume without loss of generality that $H_k = \bar{w} \cdot H_C^-$, and put $\Sigma_k = \bar{w} \cdot \Sigma_C$. Obviously, if v is a vertex of H_k not belonging to Σ_k then the link $\text{Lk}(v, H_k)$ coincides with the link $\text{Lk}(v, \Sigma')$. If $v \in \Sigma_k$ then the pair $(\text{Lk}(v, \Sigma'), \text{Lk}(v, H_k))$ is isomorphic to the pair $(L', \bar{\Omega})$, where $\bar{\Omega}$ is the subcomplex of L' equal to the closure of the connected component of $L' \setminus L$ bounded by

C . Thus, in order to prove the claim, we need to show that the subcomplex $\overline{\Omega}$ is 3-convex in L' .

To prove the latter statement, note first that since C is a full subcomplex of L' , the same is true for $\overline{\Omega}$. To get 3-convexity of $\overline{\Omega}$, consider any combinatorial geodesic (x, y, z) in the 1-skeleton of L' , with x and z belonging to $\overline{\Omega}$. We need to show that y belongs to $\overline{\Omega}$. This is obviously true if some of the vertices x, z does not lie on the boundary cycle C . So we may assume that both x and z are in C . If y is a vertex of L , it must belong to C , and hence to $\overline{\Omega}$, by 3-convexity of C in L . If y is not in L , it belongs to some component Ω' of the complement $L' \setminus L$. It follows that both x and z belong then to the closure $\overline{\Omega}'$. Since x and z are not adjacent, condition (1) of the proposition implies that $\Omega' = \Omega$, and hence again $y \in \overline{\Omega}$. This completes the proof of Claim 2.

To conclude the proof of Proposition 6.4, we now pass to estimating the diameter of the subset $\partial_\infty H_k$ with respect to some metric d_A in $\partial_\infty \Sigma' = S^{n+1}$. More precisely, put $A = 2\sqrt{n+2}$, and consider the metric d_A in $\partial_\infty \Sigma'$ as described in Lemma 6.3, for the base point v_0 . Let ξ, η be any points of $\partial_\infty H_k$, and let r_ξ, r_η be the geodesic rays in Σ' , started at v_0 , and representing points ξ, η respectively. In view of Claim 2, Proposition 5.3(a) implies that r_ξ, r_η pass through the cubical star $\text{st}_c(u_k, H_k)$. Since obviously for each $k \geq 1$ the diameter of the cubical star $\text{st}_c(u_k, H_k)$ is bounded above by A , and since the distance from v_0 to H_k is equal to t_k , we get that $d_{\Sigma'}(r_\xi(t_k), r_\eta(t_k)) \leq A$, and hence $d_A(\xi, \eta) \leq 1/t_k$. This means that for the metric d_A we have $\text{diam}(H_k) \leq 1/t_k$. The proposition follows then by Claim 1.

References

- [Ar] S. Armentrout, *Cellular decompositions of 3-manifolds that yield 3-manifolds*, Mem. Amer. Math. Soc. 107 (1971).
- [Be] M. Bestvina, *Local homology properties of boundaries of groups*, Michigan Math. J. 43 (1996), 123–139.
- [BeM] M. Bestvina, G. Mess, *The boundary of negatively curved groups*, J. Amer. Math. Soc. 4 (1991), 469–481.
- [BH] M. Bridson, A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren der mathematischen Wissenschaften 319, Springer, 1999.
- [Br] M. Brown, *Some applications of an approximation theorem for inverse limits*, Proc. Amer. Math. Soc. 11 (1960), 478–481.
- [Ca] J. Cannon, *A positional characterization of the $(n-1)$ -dimensional Sierpiński curve in S^n* , Fund. Math. 79 (1973), 107–112.
- [C] H.S.M. Coxeter, *Regular polytopes*, Methuen and Co., London, 1948.
- [DV] R. Daverman, G. Venema, *Embeddings of Manifolds*, Graduate Studies in Mathematics, vol. 106, American Mathematical Society, Providence, Rhode Island, 2009.
- [Da] M. Davis, *The geometry and topology of Coxeter groups*, London Mathematical Society Monographs Series, vol. 32, Princeton University Press, Princeton, 2008.
- [DJ] M. Davis, T. Januszkiewicz, *Hyperbolization of polyhedra*, J. Diff. Geom. 34 (1991), 347–388.
- [Dr] A. Dranishnikov, *On boundaries of hyperbolic Coxeter groups*, Topology and its Applications 110 (2001), 29–38.

- [He] J. P. Henderson, *Cellularity in polyhedra*, Topology and its Applications 12 (1981), 267–282.
- [Mo] R. L. Moore, *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc. 27 (1925), 416–428.
- [OS] D. Osajda, J. Świątkowski, *On asymptotically hereditarily aspherical groups*, Proc. London Math. Soc. 111 (2015), 93–126.
- [PS] P. Przytycki, J. Świątkowski, *Flag-no-square triangulations and Gromov boundaries in dimension 3*, Groups, Geometry & Dynamics 3 (2009), 453–468.
- [Q] F. Quinn, *Ends of maps. III: dimensions 4 and 5*, J. Diff. Geom. 17 (1982), 503–521.
- [Si] L. C. Siebenmann, *Approximating cellular maps by homeomorphisms*, Topology 11 (1972), 271–294.
- [Sw1] J. Świątkowski, *Trees of manifolds as boundaries of spaces and groups*, preprint 2013, arXiv:1304.5067.
- [Sw2] J. Świątkowski, *Hyperbolic Coxeter groups with Sierpiński carpet boundaries*, preprint 2015, arXiv:1502.02044.
- [W] D. Wise, *From Riches to Raags: 3-manifolds, Right-Angled Artin Groups, and Cubical Geometry*, CBMS Regional Conference Series in Mathematics, Number 117, AMS, 2012.